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NGUYEN NHU QUAN

SOLUTION BEHAVIOR OF FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH DELAYS

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Scientific Advisor: Assoc.Prof. PhD. Tran Dinh Ke
PhD. Nguyen Thanh Anh

Referee 1: Prof. PhD.Sci. Dinh Nho Hao, Institute of Mathematics, VAST.

Referee 2: Assoc.Prof. PhD. Nguyen Sinh Bay, Vietnam University of Commerce.

Referee 3: Assoc.Prof. PhD. Cung The Anh, Hanoi University of Education.

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INTRODUCTION

1. Motivation and outline

Fractional calculus is a very old field, dating back to a letter from Leibniz to L´Hospital in 1695. In recent decades, the subject has expanded rapidly, due to the discovery of interesting mathematical connections, and real world applications.

In history, the fractional calculus has drawn the attention from many pioneering mathematicians such as Euler, Laplace, Fourier, Liouville, Riemann, Laurant, Hardy, and Riesz etc. The applications of the fractional calculus in physics were initially undertaken by Abel and Heaviside. Nowadays, it has been seen that fractional differential equations have been widely applied in almost every scientific field, and in many realistic applications it appears to have better effects than the classical ones. Qualitative theory and its applications in physics, engineering, economics, biology and ecology are extensively discussed and demonstrated in Ahmed (2007), Butzer and Hilfer (2000), Kilbas (2006) and the references therein.

Considerable attention has been attracted to the time fractional diffusion-wave equations which arise in electromagnetic, acoustic and mechanical phenomena etc [20], and are derived from the classical diffusion or wave equations by replacing the first- or second-order time derivative by a fractional derivative of order α with $\alpha \in (0, 2]$. For $\alpha \in (0, 1]$, it is a fractional diffusion equation, which was explicitly applied to physics by Nigmatullin [24] to describe diffusion in media with fractal geometry (special types of porous media). For $\alpha \in (1, 2]$, it is a fractional wave equation, which governs the propagation of mechanical diffusive waves in viscoelatics media that reveals a power-law creep and thus provides us a physical interpretation in the framework of dynamical viscoelaticity (see Hilfer (2000), Mainardi (2001), Metzler (2000)).

On the stable in Lyapunov senses for differential inclusions, it is seen that using Lyapunov functional method is not impractical because Cauchy problems may be not uniqueness. In addition, for fractional differential systems, it is difficult to calculus the fractional derivative of Lyapunov function. Therefore, results on stability solutions in Lyapunov senses are less known. Our next objective is to use fixed point approaches introduced by Bu (2006) to analyze long time behavior for semi linear differential inclusions with infinite delays in Banach spaces, and the weak stability of zero solution will be considered.

In other direction, although the long time behaviors play an important roles in many real problems, transient behaviours are likely to be more relevant in some model related to biochemical networks or signal transduction in which processes happen in short time. It leads to the fact that *finite time dynamical system* has been studied by many mathematicians. Readers can see some recent works on this direction in the works written by Berger (2011), Duc et al. (2008, 2011, 2016),

Giesl and Rasmussen (2012), Rasmussen (2007). Using a concept of finite time dynamical for nonautonomous system in finite dimensional spaces was proposed by Giesl and Rasmussen (2012), we study the finite time attractivity for evolution differential inclusions with delays in Banach spaces. Our purposes is to look for admissible assumptions on the linear part as well as the nonlinear part such that it is possible to give sufficient conditions ensuring the finite time attractivity for the trivial solution.

We now give a brief description for our main contents in this thesis.

Firstly, we study the existence and stability of solutions to the following problem in a Banach space X:

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds + f(t, u(t), u_t),\ t > 0, t \neq t_k, k \in \Lambda, \tag{1}$$

$$u(t_k^+) - u(t_k^-) = I_k(u(t_k)), k \in \Lambda, \tag{2}$$

$$u(\bar{s}) + g(u)(\bar{s}) = \varphi(\bar{s}), \bar{s} \in [-h, 0], \tag{3}$$

where A is a closed, linear and unbounded operator, f, g and I_k are the functions which will be specified later. Here $\alpha \in (1, 2)$ and $\Lambda \subset \mathbb{N}$ is an index set. Here by $u(t_k^+)$ and $u(t_k^-)$ we mean the right and left limit of u at t_k, respectively; u_t stands for the history of the state function up to the time t, i.e. $u_t(\bar{s}) = u(t + \bar{s}), \bar{s} \in [-h, 0]$. It is obvious that, the principal part

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds$$

can be viewed as

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s)ds + u(0). \tag{4}$$

The last equation in the case A is the Laplace operator was studied in Fujita (2009). As discussed in these papers, (4) is intermediate between the diffusion ($\alpha = 1$) and the wave ($\alpha = 2$) equation. In addition, it is known that the following prototype of (1):

$$\frac{\partial u}{\partial t}(x, t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \Delta_x u(x, s)ds + F(x, t, u(x, t))$$

describes the anomalous diffusion processes and wave propagations in viscoelastic materials.

The generalized Cauchy problem involving nonlocal and/or impulsive conditions has been an active subject for many investigations in recent years. It is known that, nonlocal conditions give a better description for real models than classical initial ones, e.g., the condition

$$u(s) + \sum_{i=1}^{M} c_i u(\tau_i + s) = \varphi(s)$$

allows taking additional measurements instead of solely initial datum. The first result and physical meaning for nonlocal problems go back to the work of Byszewski

(1991). It then has aroused an increasingly interest in various nonlocal problems involving integer order differential equations and inclusions. For some remarkable solvability results, we quote here the works in Chuong (2012), Hernández (2003), Jesús (2008), Ke (2012), Liu (2003). On the other hand, impulsive conditions has been used to describe the dynamical systems having abrupt changes. A comprehensive investigation for impulsive differential equations can be found in Lakshmikantham (1989). Evidently, the generalized Cauchy problems with nonlocal conditions and impulsive effects play an important role in describing many real world problems. Due to the application of fractional derivatives in modeling and the development of fractional calculus, the integer order differential systems have been generalized to many models involving fractional differential equations.

Recently, some authors have already drawn attention to the Cauchy problems driven by fractional integro-differential equations as in (1). The results on asymptotically periodic solutions were obtained in Andrade (2010) Cuevas (2009, 2010). Considering a fractional integro-differential equation in neutral form, the authors in Caravalho (2010) showed the existence of asymptotically almost automorphic solutions. Let us take a note that in these works, the models under investigation have neither nonlocal nor impulsive condition. We also concern with the existence results in Wang (2010), in which the considered model involved a nonlocal condition. It should be noted that, in the mentioned works, no attempt has been made to consider stability problems. This is the main motivation for our study in the present paper.

Regarding the stability for differential equations, the Lyapunov functional method is an effective tool for problems in finite dimensional spaces. However, it is not easily done for fractional integro-differential equations in Banach spaces. In this work, we will employ the fixed point approach initiated by Burton and Furumochi for ordinary/functional differential equations. The main idea of this method is to construct a *stable subset*, in which the solution operator has a unique fixed point. By this approach, we will prove that the zero solution for (1)-(3) is BI-asymptotically stable, that is $u(t) \to 0$ as $t \to +\infty$ for all bounded initial data φ.

Second, we consider following problem

$$u'(t) \in \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds + F(t, u_t), t > 0, \tag{5}$$

$$u_0 = \varphi \in \mathcal{B}, \tag{6}$$

where the unknown function u takes values in a Banach space $(X, \|\cdot\|)$, A is a closed, linear and unbounded operator, F is a multi-valued map which will be specified in Section 3, $\mathcal{B}$ is an admissible phase space that will be defined later. Here $\alpha \in (1, 2)$ and u_t stands for the history of the state function up to the time t, i.e. $u_t(s) = u(t+s), s \leq 0$.

Our problem in this part involves the multi-valued nonlinearity, which derives from control systems with multi-valued feedback (Kamenskii (2001)), and various problems such as regularizing differential equations with discontinuous right-hand side (Filippov (1988)), converting differential variational inequalities (Pang (2008)). The main aim of this work is to address, for the first time, weak stability of solutions to (5)-(6). We adopt the following concept of weakly asymp-

totic stability. Let $\Sigma(\varphi)$ be the solution set of (5)-(6) with respect to the initial datum φ. Assume that $0 \in \Sigma(0)$, that is (5) admits zero solution. The zero solution of (5) is said to be weakly asymptotically stable if

1) It is stable, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\varphi|_{\mathcal{B}} < \delta$ then $|u_t|_{\mathcal{B}} < \varepsilon$ for all $u \in \Sigma(\varphi)$;
2) It is weakly attractive, i.e. for each $\varphi \in \mathcal{B}$, there exists $u \in \Sigma(\varphi)$ such that $|u_t|_{\mathcal{B}} \to 0$ as $t \to \infty$.

Let us give a short description for our approach. By analyzing the behavior of α-resolvent $\{S_\alpha(t)\}_{t\geq 0}$ generated by the linear part, we construct appropriate solution spaces, on which the solution operator has a fixed point. To this end, we define a suitable measure of noncompactness (MNC) on the solution spaces, in which the solution operator is condensing. It is worth mentioning that the fixed point approach for study stability of ordinary/functional differential equations was introduced by Burton and Furumochi as an alternative for the Lyapunov functional method.

The last content in this thesis is devoted to study finite time attractivity of the zero solution to the following functional differential system. Let $(X, \|\cdot\|)$ be a Banach space. We consider the following problem

$$D^{\alpha,\sigma}u(t) = Au(t) + f(t, u_t),\ t \in [0, T], \tag{7}$$
$$u(s) = \varphi(s), s \in [-h, 0], \tag{8}$$
$$u'(0) = y, \tag{9}$$

where $\alpha \in (1, 2), \sigma > 0$, $D^{\alpha,\sigma}$ represents the tempered fractional derivative of order α in the Caputo sense, the state function u takes values in X with the history state $u_t \in C([-h, 0]; X)$ defined by $u_t(s) = u(t + s), s \in [-h, 0]$, A is a closed linear operator on X such that $-A$ is sectorial, and the nonlinearity function f is defined on $[0, T] \times C([-h, 0]; X)$.

In the case $\sigma = 0$ and A stands for partial differential operators in a specific setting, equation (7) is used to interpret the propagation of waves in viscoelastic media. This equation in various settings has been the subject for a numerous studies in the last decades, in which the analysis of solvability, regularity and numerical schemes was addressed (see, for instance, Chen (2017), Deng (2017), Li (2016)). It should be mentioned that, the appearance of the delayed term in (7) is a nature of models derived from real life problems.

As a generalization of fractional calculus, the tempered fractional calculus has been developed to deal with a number of problems in poroelasticity (Hanyga (2001)), geophysical flows (Meerschaert (2014)), ground water hydrology (Meerschaert (2008)), etc. Many important motivations to extend the fractional calculus to tempered fractional calculus can be found in Sabzikar (2015).

Recently, there appear several works devoted to numerical methods for tempered fractional differential equations. Nevertheless, the analysis of behavior of solutions to this class of equations is less known. In particular, no attempt has been made for the qualitative study on (7). In this paper, we will prove a solvability result for problem (7)-(9) under general settings (without Lipschitz condition

on f and compactness on the semigroup generated by A, these cases were discussed in Feng (2016) for $\sigma = 0$). In addition, we analyze behavior of solutions to (7) by employing the concept of finite-time attractivity proposed in Giesl (2012).

Let us give a short description on our approach. We prove the existence result by using the fixed point theory for condensing maps, which requires the nonlinearity function f satisfy a regular property expressed by the Hausdorff measure of noncompactness. It should be noted that, in our setting, the function f may have a superlinear growth. To analyze the finite-time attractivity of solutions, we derive some local estimates (estimates with small initial data) and make use of the singular Gronwall inequality.

2. Purpose, objects and scope of the thesis

2.1. Purpose: The thesis focus on studying behavior of solutions to nonautonomuous semilinear differential evolution equations and inclusions with delays in Banach spaces using stability theory. More precisely as follows

1) The asymptotic stability of a class of fractional integro-differential equations involving impulsive effects and nonlocal conditions, whose principal part is of diffusion-wave type.
2) The existence of decay solutions and the weak asymptotic stability of the zero solution for integro-differential inclusions of diffusion-wave type involving infinite delays.
3) Establish sufficient conditions for finite-time attractivity solutions and applied to the solution of semilinear tempered fractional wave equations with sectorial operator and superlinear nonlinearity..

2.2. Objects: In the thesis, we consider some types of semilinear fractional differential systems in Banach spaces:

- The first type: Fractional integro-differential equations involving impulsive effects and nonlocal conditions with finite delays;
- The second one: Fractional integro-differential inclusions with infinite delays;
- The third one: Semilinear tempered fractional wave equations with finite delays.

2.3. Scope: The scope of the thesis is defined by the following contents

- **Content 1:** Study the solvability for nonautonomuos integro-differential equations and inclusions with delays;
- **Content 2:** Study the asymptotic stability for the zero solution to fractional integro-differential equations involving impulsive effects and nonlocal conditions with finite delays;
- **Content 3:** Study the weak asymptotic stability for the zero solution to fractional integro-differential inclusions with infinite delays;
- **Content 4:** Study the finite-time attractivity for the solution to semilinear tempered fractional wave equations with finite delays.

3. Research Methods

- To prove the existence of solutions to fractional differential equations and inclusions, we employ the semigroup theory (see MNC's estimates (see Bothe (1998)), and fixed point theory for condensing multimaps (see Kamenskii et al (2001)).
- To study the stability of the zero solution to fractional differential equations and inclusions with delays and the finite-time attractivity of the zero solution to system with finite delays, we make use of the fixed point principles and prior estimates techniques.

4. Structure and Results

Together with the Introduction, Conclusion, Author's works related to the thesis that have been published and References, the thesis includes five chapters: Chapter 1 is devoted to present some preliminaries. In Chapter 2, we present the solvability and prove the asymptotic stability for the zero solution to fractional integro-differential equations involving impulsive effects and nonlocal conditions with finite delays. Chapter 3 we study the weak asymptotic stability for the zero solution to fractional integro-differential inclusions with infinite delays. Chapter 4 we prove the finite-time attractivity for the solution to semilinear tempered fractional wave equations with finite delays on compact intervals.

Chapter 1

PRELIMINARIES

In this chapter, we present some preliminaries including: some results about fractional caculus; resolvent theory; measure of noncompactness; multi-valued calculus and fixed point principles.

1.1. FRACTIONAL CACULUS

In this section, we recall some concept and properties related to fractional derivatives and fractional integral.

1.2. RESOLVENT THEORY

In this section, we recall some concept related to cosine family, semigroup and resolvent.

1.3. MEASURE OF COMPACTNESS (MNC) AND MNC ESTIMATES

In this section, we present concept of MNC and some estimates related to Hausdorff MNC. It is noted that some estimates on Banach space and separable Banach space are different.

1.4. MULTIVALUED CALCULUS AND SOME FIXED POINT THEOREMS

1.4.1. Multivalued calculus

In this subsection, we present some definitions and results in multivalued calculus, including concept of selector and the existence of selection function. Some important results are stated in Proposition 1.5 and 1.6.

1.4.2. Condensing map and some fixed point theorems

In this subsection, we recall some fixed point principles for condensing multivalued map.

1.5. STABILITY FOR DIFFERENTIAL SYSTEMS

In this section, we recall some concept related to Lyapunov stability, weak stability, finite-time stability and finite-time attractivity.

1.6. FUNCTIONAL SPACES

In this section, we recall some functional spaces used in the thesis, such as: $L^p(\Omega), 1 \leq p < +\infty$; $L^\infty(\Omega)$; $L^p_{loc}(\Omega), 1 \leq p < +\infty$, where Ω a bounded domain in $\mathbb{R}^n$. some functional spaces depending on time: $C([a,b];E)$; $L^p(a,b;E)$; $\mathcal{C}_\tau := C([-\tau,0];E)$, $\tau > 0$ given, where E is a Banach space. Specially, we recall the phase space introduced by Hale-Kato $\mathcal{B}$.

Chapter 2

STABILITY FOR A CLASS OF SEMILINEAR DEFFUSION-WAVE EQUATIONS WITH IMPULSIVE CONDITION AND FINITE DELAYS

In this chapter, we deal with a class of fractional integro-differential equations involving impulsive effects and nonlocal conditions, whose principal part is of diffusion-wave type. Our aim is to establish some existence and stability results for integral solutions to the mentioned problem by fixed point approach.

The content of this chapter is written based on the paper [1] in the author's works related to the thesis that has been published.

2.1. PROBLEM SETTING

Let $(X, \|\cdot\|)$ be a Banach space, we consider the following problem:

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds + f(t, u(t), u_t),\ t > 0, t \neq t_k, k \in \Lambda, \tag{2.1}$$

$$u(t_k^+) - u(t_k^-) = I_k(u(t_k)), k \in \Lambda, \tag{2.2}$$

$$u(\bar{s}) + g(u)(\bar{s}) = \varphi(\bar{s}), \bar{s} \in [-h, 0], \tag{2.3}$$

where A is a closed, linear and unbounded operator, f, g and I_k are the functions which will be specified later. Here $\alpha \in (1, 2)$ and $\Lambda \subset \mathbb{N}$ is an index set. Here by $u(t_k^+)$ and $u(t_k^-)$ we mean the right and left limit of u at t_k, respectively; u_t stands for the history of the state function up to the time t, i.e. $u_t(\bar{s}) = u(t + \bar{s}), \bar{s} \in [-h, 0]$.

2.2. SOLVABILITY

In this section, let $\mathcal{C}_h = C([-h, 0]; X)$ and χ_h be the Hausdorff MNC in $\mathcal{C}_h$. Concerning problem (1)-(3), we give the following assumptions.

(**A**) *The operator A is sectorial of type (ω, θ) with $0 \leq \theta < \pi(1 - \alpha/2)$ so that the α-resolvent $S_\alpha(\cdot)$ generated by A is norm continuous for $t > 0$.*

(**F**) *The nonlinear function $f : \mathbb{R}^+ \times X \times \mathcal{C}_h \to X$ satisfies:*

(1) $f(\cdot, v, w)$ is measurable for each $(v, w) \in X \times \mathcal{C}_h$, $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0, T]$ and

$$||f(t, v, w)||_X \leq k(t)\Psi_f(||v||_X + ||w||_{\mathcal{C}_h}),$$

for all $(v, w) \in X \times \mathcal{C}_h$, where $k \in L^1_{loc}(\mathbb{R}^+)$, Ψ_f is a real-valued, continuous and nondecreasing function;

(2) there exists a function $m : \mathbb{R}^2_+ \to \mathbb{R}^+$ such that $m(t, \cdot) \in L^1(0, t), t > 0$, and for all bounded subsets $V \subset X, W \subset \mathcal{C}_h$,

$$\chi(S_\alpha(t-s)f(s, V, W)) \leq m(t, s)[\chi(V) + \chi_h(W)],$$

for a.e. $t, s \in [0,T], s \leq t$.

(**I**) *The impulsive function* $I_k : X \to X, k \in \Lambda$, *satisfies:*

(1) I_k *is continuous and there exists* $l_k \geq 0$ *verifying that*

$$||I_k(x)|| \leq l_k \Psi_I(||x||),$$

where Ψ_I *is a real-valued, continuous and nondecreasing function;*

(2) there is a number $\mu_k \geq 0$ *such that*

$$\chi(I_k(V) \leq \mu_k \chi(V),$$

for all bounded set $V \subset X$.

(**G**) *The nonlocal function* $g : \mathcal{PC}([-h,T];X) \to \mathcal{C}_h$ *obeys the following conditions:*

(1) g *is continuous and*

$$||g(u)||_{\mathcal{C}_h} \leq \Psi_g(||u||_{\mathcal{PC}}),$$

for all $u \in \mathcal{PC}$, *where* Ψ_g *is a continuous and nondecreasing function on* $\mathbb{R}^+$;

(2) there exists $\eta \geq 0$ *such that for any bounded set* $D \subset \mathcal{PC}([-h,T];X)$,

$$\chi_h(g(D)) \leq \eta \chi_{\mathcal{PC}}(D).$$

We have the following definition of integral solution to (2.1)-(2.3).

Definition 2.1. A function $u \in \mathcal{PC}([-h,T];X)$ is said to be an integral solution of problem (2.1)-(2.3) on the interval $[-h,T]$ if and only if $u(t) = g(u)(t) + \varphi(t)$ for $t \in [-h,0]$, and

$$u(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(u(t_k)) + \int_0^t S_\alpha(t-s) f(s,u(s),u_s) ds,$$

for any $t \in [0,T]$.

Let $\mathcal{F} : \mathcal{PC}([-h,T];X) \to \mathcal{PC}([-h,T];X)$, where

$$\mathcal{F}(u)(t) = \begin{cases} \varphi(t) + g(u)(t), & t \in [-h,0], \\ S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum\limits_{0<t_k<t} S_\alpha(t-t_k) I_k(u(t_k)) & \\ \qquad + \int_0^t S_\alpha(t-s) f(s,u(s),u_s) ds, & t \in [0,T]. \end{cases}$$

Then u is an integral solution of (2.1)-(2.3) if it is a fixed point of *the solution operator* $\mathcal{F}$.

Now we prove the condensivity of the solution operator.

Lemma 2.1. *Let the hypotheses* (**A**), (**F**), (**G**) *and* (**I**) *hold. Then the solution operator* $\mathcal{F}$ *satisfies*

$$\chi_{\mathcal{PC}}(\mathcal{F}(D)) \leq \left[\Big(\eta + \sum_{t_k \in [0,T]} \mu_k \Big) S_\alpha^T + 8 \sup_{t \in [0,T]} \int_0^t m(t,s) ds \right] \chi_{\mathcal{PC}}(D),$$

for all bounded set $D \subset \mathcal{PC}([-h,T],X)$, *here* $S_\alpha^T = \sup_{t \in [0,T]} ||S_\alpha(t)||$.

Using the fixed point theorem for condensing operators gives us the existence result.

Theorem 2.1. *Assume that the hypotheses of Lemma 2.1 hold. Then the problem* (2.1)-(2.3) *has at least one integral solution in* $\mathcal{PC}([-h,T];X)$*, provided that*

$$\Big(\eta+\sum_{t_k\in[0,T]}\mu_k\Big)S_\alpha^T+8\sup_{t\in[0,T]}\int_0^t m(t,s)ds<1, \tag{2.4}$$

and

$$\begin{aligned}\liminf_{r\to\infty}\frac{1}{r}\Bigg[\Big(\Psi_g(r)+\Psi_I(r)\sum_{t_k\in[0,T]}l_k\Big)S_\alpha^T\\ +\Psi_f(2r)\sup_{t\in[0,T]}\int_0^t\|S_\alpha(t-s)\|k(s)ds\Bigg]<1,\end{aligned} \tag{2.5}$$

where S_α^T *is given in Lemma 2.1.*

2.3. STABILITY RESULTS

In order to study the stability results for problem (2.1)-(2.3), we consider the function space

$$\mathcal{PC}_0=\{u\in\mathcal{PC}([-h,+\infty);X):\lim_{t\to\infty}u(t)=0\}$$

with the norm

$$\|u\|_\infty=\sup_{t\geq 0}\|u(t)\|,$$

where $\mathcal{PC}([-h,\infty);X)$ is defined similarly to $\mathcal{PC}([-h,T];X)$ as $T=+\infty$.

Then $\mathcal{PC}_0$ is a Banach space. In this section, we replace assumptions (**A**), (**F**), (**G**) and (**I**) by the following:

(**Aa**) *The operator* A *is sectorial of type* (ω,θ) *such that* $\omega<0$ *and* $0\leq\theta<\pi(1-\alpha/2)$.

(**Fa**) $f(\cdot,v,w)$ *is measurable for each* $v\in X, w\in\mathcal{C}_h$, $f(t,\cdot,\cdot)$ *is continuous for a.e.* $t\in\mathbb{R}^+$, $f(t,0,0)=0$, *and there exists* $k\in L^1(\mathbb{R}^+)$, *such that*

$$||f(t,v_1,w_1)-f(t,v_2,w_2)||\leq k(t)(||v_1-v_2||+||w_1-w_2||_{\mathcal{C}_h}), t\in\mathbb{R}^+,$$

for all $v_1,v_2\in X, w_1,w_2\in\mathcal{C}_h$.

(**Ga**) g *is a continuous function satisfying that* $g(0)=0$ *and there is a nonnegative number* η *such that*

$$||g(w_1)-g(w_2)||_{\mathcal{C}_h}\leq\eta||w_1-w_2||_{\mathcal{PC}},$$

for all $w_1,w_2\in\mathcal{PC}([-h,T];X)$, *with all* $T>0$.

(**Ia**) $I_k, k\in\Lambda$, *is continuous,* $I_k(0)=0$ *and there exist a sequence* $\{\mu_k\}, k\in\Lambda$ *such that* $\sum_{k\in\Lambda}\mu_k<\infty$, *and*

$$||I_k(x)-I_k(y)||\leq\mu_k||x-y||,\ \text{for all } x,y\in X.$$

Theorem 2.2. *Let* (**Aa**), (**Fa**), (**Ga**), *and* (**Ia**) *hold. Then problem* (2.1)-(2.3) *has a unique solution* $u \in \mathcal{PC}_0$ *provided that*

$$\Big(\eta + \sum_{k\in\Lambda} \mu_k\Big) S_\alpha^\infty + 2\sup_{t\geq 0} \int_0^t \|S_\alpha(t-s)\| k(s) ds < 1, \tag{2.6}$$

where $S_\alpha^\infty = \sup_{t\geq 0} \|S_\alpha(t)\|$.

2.4. APPLICATION

We give an application to the abstract results. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary $\partial\Omega$. We consider the following system:

$$\frac{\partial u}{\partial t}(x,t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} L_x u(x,s) ds + k(t)\tilde{f}(x, u(x,t), u(x,t-h)), \tag{2.7}$$

$$\alpha \in (1,2), t \in \mathbb{R}^+ \backslash \{t_k\}_{k\in\Lambda}, x \in \Omega,$$

$$u(x,t) = 0,\ x \in \partial\Omega, t \in \mathbb{R}^+, \tag{2.8}$$

$$u(x,t_k^+) = u(x,t_k) + \tilde{I}_k(x, u(x,t_k)), x \in \Omega, k \in \Lambda, \tag{2.9}$$

$$u(x,s) + \sum_{i=1}^M c_i u(x, \tau_i + s) = \varphi(s), s \in [-h, 0], x \in \Omega, \tag{2.10}$$

where the operator

$$L_x = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

has the property

$$\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \theta|\xi|^2,\ \forall \xi \in \mathbb{R}^n$$

with $\theta > 0$. Let $X = L^2(\Omega), A = L_x$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then system (2.7)-(2.10) is in the form of the abstract model (2.1)-(2.3) with

$$f(t,v,w)(x) = k(t)\tilde{f}(x, v(x), w(x,-h)), v \in X, w \in C([-h,0];X),$$

$$I_k(v)(x) = \tilde{I}_k(x, v(x)), v \in X,$$

$$g(u)(s)(x) = \sum_{i=1}^M c_i u(x, \tau_i + s), u \in \mathcal{PC}([-h,+\infty);X).$$

It is known that, A is a sectorial operator and it generates an analytic semigroup in X. Moreover, one can check that A is sectorial of type $(\lambda_1, 0)$ where $\lambda_1 < 0$ is the first eigenvalue of A.

Assume that $k \in L^1(\mathbb{R})$ and $\tilde{f} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$|\tilde{f}(x,y_1,z_1) - \tilde{f}(x,y_2,z_2)| \leq \kappa(x)(|y_1 - y_2| + |z_1 - z_2|), \kappa \in X,$$

for all $x \in \Omega, y_1, y_2, z_1, z_2 \in \mathbb{R}$. Then we have

$$\begin{aligned} ||f(t,v_1,w_1) - f(t,v_2,w_2)|| &\leq k(t)||\kappa||(||v_1 - v_2|| + ||w_1(\cdot,-h) - w_2(\cdot,-h)||) \\ &\leq k(t)||\kappa||(||v_1 - v_2|| + ||w_1 - w_2||_{\mathcal{C}_h}), \end{aligned}$$

for all $v_1, v_2 \in X, w_1, w_2 \in \mathcal{C}_h$.

Let $\tilde{I}_k : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that

$$|\tilde{I}_k(x, y_1) - \tilde{I}_k(x, y_2)| \leq \ell_k(x)|y_1 - y_2|, \ell_k \in X,$$

for all $x \in \Omega, y_1, y_2 \in \mathbb{R}$. Then

$$||I_k(v_1) - I_k(v_2)|| \leq ||\ell_k||\ ||v_1 - v_2||, \forall v_1, v_2 \in X.$$

Regarding the nonlocal function g, it is obvious that

$$||g(u_1) - g(u_2)||_{\mathcal{C}_h} \leq \Big(\sum_{i=1}^{M} c_i\Big)||u_1 - u_2||_{\mathcal{PC}}, \forall u_1, u_2 \in \mathcal{PC}([-h, T]; X), \forall T > 0.$$

Under the above settings, applying Theorem 2.2, one can state that problem (2.7)-(2.10) has a unique integral solution in $\mathcal{PC}_0$, provided that

$$\Big(\sum_{i=1}^{M} c_i + \sum_{k \in \Lambda} ||\ell_k||\Big) S_\alpha^\infty + 2||\kappa|| \sup_{t \geq 0} \int_0^t \|S_\alpha(t-s)\| k(s) ds < 1.$$

Chapter 3

WEAK STABILITY FOR A CLASS OF DEFFUSION-WAVE INCLUSIONS WITH INFINITE DELAYS

In this chapter, we deal with the Cauchy problem associated with integro-differential inclusions of diffusion-wave type involving infinite delays. Based on the behavior of resolvent operator associated with the linear part, an explicit estimate for solutions will be established. As a consequence, the weak stability of zero solution is proved in case the resolvent operator is asymptotically stable.

The content of this chapter is written based on the paper [2] in the author's works related to the thesis that has been published.

3.1. PROBLEM SETTING

We consider the following problem

$$u'(t) \in \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds + F(t, u_t), t > 0, \tag{3.1}$$

$$u_0 = \varphi \in \mathcal{B}, \tag{3.2}$$

where the unknown function u takes values in a Banach space $(X, \|\cdot\|)$, A is a closed, linear and unbounded operator, F is a multi-valued map which will be specified in Section 3, $\mathcal{B}$ is an admissible phase space that was defined in Chapter 1. Here $\alpha \in (1, 2)$ and u_t stands for the history of the state function up to the time t, i.e. $u_t(s) = u(t+s), s \leq 0$.

3.2. EXPONENTIALLY BOUNDED SOLUTIONS

Concerning problem (3.1)-(3.2), we give the following assumptions:

(**A**) The operator A is sectorial of type (ω, θ) with $\omega \geq 0$ and $0 \leq \theta < \pi(1-\alpha/2)$ so that the α-resolvent $S_\alpha(\cdot)$ generated by A is norm continuous.

(**B**) The phase space $\mathcal{B}$ obeys (B1)-(B4) such that the functions K and $\frac{M}{g}$ are uniformly bounded.

(**F**) The multimap $F : \mathbb{R}^+ \times \mathcal{B} \to Kv(X)$ satisfies:

(1) for any $\psi \in \mathcal{B}$ the multimap $F(\cdot, \psi) : \mathbb{R}^+ \to Kv(X)$ admits a locally strongly measurable selector, i.e. for each $T > 0$ one can find a strongly measurable function $f : [0, T] \to X$ such that $f(t) \in F(t, \psi)$ for a.e. $t \in [0, T]$;

(2) for a.e. t $\in \mathbb{R}^+$ the multimap $F(t, \cdot) : \mathcal{B} \to Kv(X)$ is u.s.c on $\mathcal{B}$;

(3) there exist nonnegative functions m, p such that $m, \frac{p}{g} \in L^1(\mathbb{R}^+)$, and for every $\psi \in \mathcal{B}$ we have

$$\|F(t, \psi)\| := \sup\{\|\xi\| : \xi \in F(t, \psi)\} \leq m(t)|\psi|_{\mathcal{B}} + p(t)$$

for a.e. $t \in \mathbb{R}^+$;

(4) there exists a function $k \in L^\infty(\mathbb{R}^+)$ such that, for every bounded set $D \subset \mathcal{B}$ we have $\chi(F(t,D)) \leq k(t) \sup_{s \leq 0} \chi(D(s))$ a.e. $t \in \mathbb{R}^+$.

By the arguments in Chapter 2, we adopt the following definition of solution to (3.1)-(3.2).

Definition 3.1. Given $\varphi \in \mathcal{B}$. A function $u : \mathbb{R} \to X$ is said to be an integral solution of problem (3.1)-(3.2) if there exists $f \in \mathcal{P}_F(u|_{\mathbb{R}^+})$ such that

$$u(t) = \begin{cases} \varphi(t), & t \leq 0, \\ S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(t-s)f(s)ds, & t > 0. \end{cases}$$

Now we consider the multi-valued operator

$$\begin{aligned} &\mathcal{F} : \mathbf{C}_{g,\varphi} \to \mathcal{P}(\mathbf{C}_{g,\varphi}), \\ &\mathcal{F}(v)(t) = S_\alpha(t)\varphi(0) + \left\{ \int_0^t S_\alpha(t-s)f(s)ds : f \in \mathcal{P}_F(v) \right\} \end{aligned} \tag{3.3}$$

It is clear that if v is a fixed point of $\mathcal{F}$ then $v[\varphi]$ is an integral solution to (3.1)-(3.2).

Theorem 3.1. *Let the assumptions (**A**), (**B**) and (**F**) hold. Then the problem* (3.1)-(3.2) *has at least one integral solution satisfying* $\frac{\|u(t)\|}{g(t)} = o(1)$ *as* $t \to \infty$, *provided that*

$$\sup_{t \geq 0} \int_0^t \frac{\|S_\alpha(t-s)\|}{g(t-s)} K(s)m(s)ds < 1. \tag{3.4}$$

3.3. WEAK STABILITY RESULT

One observes that, by choosing $g \equiv 1$ and proceeding as in the previous section, we can prove the existence of attracting solutions to (3.1)-(3.2) and then infer the weakly asymptotic stability of zero solution. Precisely, we consider the solution operator $\mathcal{F}$ on the space

$$BC_{0,\varphi} = \{v \in C([0,\infty); X) : v(0) = \varphi(0),\ \lim_{t\to\infty} \|v(t)\| = 0\},$$

with the norm

$$\|v\|_\infty = \sup_{t \geq 0} \|v(t)\|.$$

In this circumstance, (**A**), (**B**) and (**F**) are replaced by the following assumptions.

(**A**') The operator A is sectorial of type (ω, θ) with $\omega < 0$ and $0 \leq \theta < \pi(1-\alpha/2)$ so that the α-resolvent $S_\alpha(\cdot)$ generated by A is norm continuous.

(**B**') The phase space $\mathcal{B}$ obeys (B1)-(B4) such that the function K is uniformly bounded and M satisfies $M(t) = o(1)$ as $t \to \infty$.

(**F**') The multimap $F : \mathbb{R}^+ \times \mathcal{B} \to Kv(X)$ satisfies:

(1) for any $\psi \in \mathcal{B}$ the multimap $F(\cdot, \psi) : \mathbb{R}^+ \to Kv(X)$ admits a locally strongly measurable selector, i.e. for each $T > 0$ one can find a strongly measurable function $f : [0, T] \to X$ such that $f(t) \in F(t, \psi)$ for a.e. $t \in [0, T]$;

(2) for a.e. t $\in \mathbb{R}^+$ the multimap $F(t, \cdot) : \mathcal{B} \to Kv(X)$ is u.s.c on $\mathcal{B}$;

(3) there exists a function $m \in L^1(\mathbb{R}^+)$ such that for every $\psi \in \mathcal{B}$ we have

$$\|F(t, \psi)\| := \sup\{\|\xi\| : \xi \in F(t, \psi)\} \leq m(t)|\psi|_{\mathcal{B}},$$

for a.e. $t \in \mathbb{R}^+$;

(4) there exists a function $k \in L^\infty(\mathbb{R}^+)$ such that, for every bounded set $D \subset \mathcal{B}$ we have $\chi(F(t, D)) \leq k(t) \sup_{s \leq 0} \chi(D(s))$ a.e. $t \in \mathbb{R}^+$.

As a consequence, we have the following result.

Theorem 3.2. *Let the hypotheses (**A**'), (**B**') and (**F**') holds. If the following condition*

$$\Lambda_\infty = \sup_{t \geq 0} \int_0^t \|S_\alpha(t - s)\| K(s) m(s) ds < 1, \tag{3.5}$$

is satisfied, then the zero solution of the problem (3.1)-(3.2) *is weakly asymptotically stable.*

3.4. APPLICATION

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider the following PDE

$$\frac{\partial u}{\partial t}(t, x) = \int_0^t \frac{(t - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} \Delta_x u(s, x) ds + f(t, x), t > 0, x \in \Omega, \tag{3.6}$$

where $\alpha \in (1, 2)$, the unknown function u satisfies the boundary condition

$$u(t, x) = 0, x \in \partial\Omega, t \geq 0, \tag{3.7}$$

and the initial condition

$$u(t, x) = \varphi(t, x), x \in \Omega, t \leq 0; \tag{3.8}$$

the function f is subject to

$$f(t, x) \in b(t, x) \int_{-\infty}^0 \int_\Omega \nu(\theta, y) \big[f_1(u(t + \theta, y)), f_2(u(t + \theta, y))\big] dy d\theta, \tag{3.9}$$

here $[f_1, f_2] = \{\tau f_1 + (1 - \tau) f_2 : \tau \in [0, 1]\}$. The system (3.6)-(3.9) can be seen as a control problem with multi-valued feedback. Let $A = \Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, then the resolvent set of A, $\rho(A) \subset \mathbb{C} \backslash (-\infty, -\lambda_1]$, where $\lambda_1 = \sup\{\|\nabla u\|^2_{L^2(\Omega)} : \|u\|_{L^2(\Omega)} = 1\}$. So A is sectorial of type $(-\lambda_1, \theta)$ with any $\theta \in (0, \frac{\pi}{2})$.

Let $X = L^2(\Omega)$. One sees that $S_\alpha(\cdot)$ is differentiable on $(0, \infty)$ and

$$\left\| \frac{d}{dt} S_\alpha(t) u_0 \right\| \leq C_\alpha t^{-1} \|u_0\|, \ t > 0,$$

where C_α is a positive constant. In particular, $S_\alpha(\cdot)$ is norm continuous. So (**A**') is just fulfilled.

Consider the phase space $\mathcal{B} = CL^2_g$ with $g(\theta) = e^{h\theta}$, where the seminorm in $\mathcal{B}$ is given by

$$|w|_{\mathcal{B}} = \sup_{-r\le\theta\le 0} \|w(\theta)\|_X + \left(\int_{-\infty}^{-r} e^{h\theta}\|w(\theta)\|_X^2 d\theta\right)^{\frac{1}{2}},$$

for $r = -\frac{1}{h}\ln(1-h)$. We have

$$K(t) = \begin{cases} 1, & 0 \le t \le r, \\ 1 + \frac{1}{\sqrt{h}}\sqrt{e^{-hr} - e^{-ht}}, & t > r; \end{cases}$$

$$M(t) = \begin{cases} \max\left\{e^{-\frac{1}{2}ht}, 1 + \sqrt{\frac{e^{-hr}}{h}(1-e^{-ht})}\right\}, & 0 \le t \le r, \\ e^{-\frac{1}{2}ht}, & t > r. \end{cases}$$

So one observes that $K(t) = O(1), M(t) = o(1)$ as $t \to \infty$, and then (**B**') is satisfied.

We will show that the system (3.6)-(3.9) is weakly asymptotically stable under a suitable setting. Specifically, we assume that

(N1) $b \in L^1(\mathbb{R}^+; L^2(\Omega))$;

(N2) $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy

$$|f_i(z)| \le C_f|z|, \forall z \in \mathbb{R},$$

for some $C_f > 0$;

(N3) $\nu : (-\infty, 0] \times \Omega \to \mathbb{R}$ is a continuous function such that

$$|\nu(\theta, y)| \le C_\nu e^{h\theta}, \forall y \in \Omega, \theta \le 0,$$

where $h \in (0,1)$ and $C_\nu > 0$.

Let $F : \mathbb{R}^+ \times \mathcal{B} \to \mathcal{P}(X)$ be the multimap defined by

$$F(t,w)(x) = b(t,x)\int_{-\infty}^0 \int_\Omega \nu(\theta,y)\big[f_1(w(\theta,y)), f_2(w(\theta,y))\big]dyd\theta.$$

Then we see that, for each $\tau \in [0,1]$ the function

$$f(t,x) = b(t,x)\int_{-\infty}^0 \int_\Omega \nu(\theta,y)\big[\tau f_1(w(\theta,y)) + (1-\tau)f_2(w(\theta,y))\big]dyd\theta$$

is an integrable selector of $F(t,w)$. The assumption (**F**')(1) is just satisfied.

Now for each bounded set $W \subset \mathcal{B}$, $F(t,W) \subset \text{span}\{b(t,\cdot)\}$, which is an one-dimension subspace of $L^2(\Omega)$. In addition, by (N2) $F(t,W)$ is also bounded and then relatively compact. Hence

$$\chi(F(t,W)) = 0,$$

and the assumption (**F**')(4) is verified with $k = 0$. In particular, $F(t, w)$ is relatively compact for each $w \in \mathcal{B}$. In view of the continuity of f_1, f_2 and the Lebesgue dominated convergence theorem, we can testify the closedness of $F(t, \cdot)$. Thus $F(t, \cdot)$ has compact values and is u.s.c. The assumption (**F**')(2) is satisfied.

Regarding (**F**')(3), we have the following estimates

$$\begin{aligned}
\|F(t,w)\|^2 &\leq C_\nu^2 C_f^2 \|b(t,\cdot)\|^2 \left(\int_{-\infty}^0 e^{h\theta} \int_\Omega |w(\theta,y)| dy d\theta \right)^2 \\
&\leq \frac{1}{h} C_\nu^2 C_f^2 \|b(t,\cdot)\|^2 \int_{-\infty}^0 e^{h\theta} \left(\int_\Omega |w(\theta,y)| dy \right)^2 d\theta \\
&\leq \frac{1}{h} C_\nu^2 C_f^2 \mu(\Omega) \|b(t,\cdot)\|^2 \int_{-\infty}^0 e^{h\theta} \|w(\theta,\cdot)\|^2 d\theta \\
&= \frac{1}{h} C_\nu^2 C_f^2 \mu(\Omega) \|b(t,\cdot)\|^2 \left(\int_{-r}^0 + \int_{-\infty}^{-r} \right) e^{h\theta} \|w(\theta,\cdot)\|^2 d\theta \\
&\leq \frac{1}{h} C_\nu^2 C_f^2 \mu(\Omega) \|b(t,\cdot)\|^2 \left(\|w\|^2_{C([-r,0];X)} + \int_{-\infty}^{-r} e^{h\theta} \|w(\theta,\cdot)\|^2 d\theta \right) \\
&\leq \frac{1}{h} C_\nu^2 C_f^2 \mu(\Omega) \|b(t,\cdot)\|^2 |w|^2_{\mathcal{B}},
\end{aligned}$$

thanks to the Hölder inequality and the fact that $r = -\frac{1}{h}\ln(1-h)$, here $\mu(\Omega)$ is the Lebesgue measure of Ω. Then

$$\|F(t,w)\| \leq \frac{1}{\sqrt{h}} C_\nu C_f \sqrt{\mu(\Omega)} \|b(t,\cdot)\| |w|_{\mathcal{B}}.$$

Thus (**F**')(3) is testified with $m(t) = \frac{1}{\sqrt{h}} C_\nu C_f \sqrt{\mu(\Omega)} \|b(t,\cdot)\|$.

Finally, if we assume, in addition, that $b(t,x) = b_1(t) b_2(x)$ with $b_1 \in L^1(\mathbb{R}^+), b_2 \in L^2(\Omega)$ then

$$\|b(t,\cdot)\| = b_1(t) \|b_2\|,$$

and we can verify condition (3.5) as follows

$$\Lambda_\infty \leq \sup_{t \geq 0} \int_0^t \frac{C_0 K_\infty \|b_2\| b_1(s)}{1 + \lambda_1 (t-s)^\alpha} ds \leq C_0 K_\infty \|b_2\| \|b_1\|_{L^1(\mathbb{R}^+)},$$

where $C_0 = \frac{1}{\sqrt{h}} C C_\nu C_f \sqrt{\mu(\Omega)}$. Therefore (3.5) hold with $\|b_2\| \|b_1\|_{L^1(\mathbb{R}^+)}$ small.

Chapter 4

FINITE-TIME ATTRACTIVITY FOR SEMILINEAR TEMPERED FRACTIONAL WAVE EQUATIONS

We prove the existence and finite-time attractivity of solutions to semilinear tempered fractional wave equations with sectorial operator and superlinear nonlinearity. Our analysis is based on the α-resolvent theory, the fixed point theory for condensing maps and the local estimates of solutions. An application to a class of partial differential equations will be given.

The content of this chapter is written based on the paper [3] in the author's works related to the thesis that has been published.

4.1. PROBLEM SETTING

Let $(X, \|\cdot\|)$ be a Banach space. We consider the following problem

$$D^{\alpha,\sigma}u(t) = Au(t) + f(t, u_t),\ t \in [0, T], \tag{4.1}$$

$$u(s) = \varphi(s), s \in [-h, 0], \tag{4.2}$$

$$u'(0) = y, \tag{4.3}$$

where $\alpha \in (1, 2), \sigma > 0$, $D^{\alpha,\sigma}$ represents the tempered fractional derivative of order α in the Caputo sense, the state function u takes values in X with the history state $u_t \in C([-h, 0]; X)$ defined by $u_t(s) = u(t + s), s \in [-h, 0]$, A is a closed linear operator on X such that $-A$ is sectorial, and the nonlinearity function f is defined on $[0, T] \times C([-h, 0]; X)$.

4.2. EXISTENCE RESULTS

To study the existence of problem (4.1)-(4.3), we give the following assumptions.

(**A**) The operator $-A$ is sectorial of angle θ with $0 \le \theta < \pi(1 - \alpha/2)$.

(**F**) The map $f : [0, T] \times \mathcal{C}_h \to X$ is continuous and satisfies:

(1) the growth condition

$$\|f(t, v)\| \le a + b\|v\|_{\mathcal{C}_h}, \forall v \in \mathcal{C}_h,\ \text{for a.e. } t \in [0, T],$$

where a, b are nonnegative numbers;

(2) if $(\lambda I - A)^{-1}, \lambda \in \rho(A)$, is non-compact then for any bounded set $\Omega \subset \mathcal{C}_h$ we have

$$\chi(f(t, \Omega)) \le k(t) \sup_{\theta \in [-h,0]} \chi(\Omega(\theta)),$$

where $k \in L^1(0, T)$ is a nonnegative function.

Let $\mathcal{C}_h = C([-h,0];X)$. Then $\mathcal{C}_h \times X$ is a Banach space endowed with the norm

$$\|(x,y)\| = \sup_{s\in[-h,0]} \|x(s)\| + \|y\| = \|x\|_{\mathcal{C}_h} + \|y\|.$$

we give the following definition of integral solutions for (1)-(3).

Definition 4.1. Given $(\varphi, y) \in \mathcal{C}_h \times X$. A function $u \in C([-h,T];X)$ is said to be an integral solution of problem (1)-(3) on the interval $[-h,T]$ if and only if $u(t) = \varphi(t)$ for $t \in [-h,0]$ and

$$\begin{aligned} u(t) = e^{-\sigma t} S_\alpha(t)\varphi(0) + e^{-\sigma t}(1 * S_\alpha)(t)(y + \sigma\varphi(0))ds, \\ + \int_0^t e^{-\sigma(t-s)}(g_{\alpha-1} * S_\alpha)(t-s) f(s,u_s)ds, \end{aligned}$$

for any $t \in [0,T]$.

For $\varphi \in \mathcal{C}_h$, we define the space

$$C_\varphi = \{u \in C([0,T];X) : u(0) = \varphi(0)\}.$$

as a closed subspace of $C([0,T];X)$. If $v \in C_\varphi$, we have the function $v[\varphi] : [-h,T] \to X$ defined by

$$v[\varphi](t) = \begin{cases} \varphi(t) & \text{if } -h < t \le 0, \\ v(t) & \text{if } t \in [0,T]. \end{cases}$$

Then, clearly

$$v[\varphi]_t(\theta) = \begin{cases} \varphi(t+\theta), & -h-t < \theta < -t, \\ v(t+\theta), & \theta \in [-t,0]. \end{cases}$$

Now we consider the operator $\mathcal{F} : C_\varphi \to C_\varphi$ given by

$$\begin{aligned} \mathcal{F}(v)(t) = e^{-\sigma t} S_\alpha(t)\varphi(0) + e^{-\sigma t}(1 * S_\alpha)(t)(y + \sigma\varphi(0)) \\ + \int_0^t e^{-\sigma(t-s)}(g_{\alpha-1} * S_\alpha)(t-s) f(s,v_s)ds. \end{aligned} \tag{4.4}$$

It is clear that if v is a fixed point of $\mathcal{F}$ then $v[\varphi]$ is an integral solution to (4.1)-(4.3).

Theorem 4.1. *Assume that the hypotheses* $(\boldsymbol{A})$ *and* $(\boldsymbol{F})$ *hold. Then the solution set of the problem* (4.1)-(4.3) *is nonempty and compact.*

The assumption $(\mathbf{F})(1)$ indicate that f has a sublinear growth. We now consider the case f is possibly superlinear, so we replace $(\mathbf{F})$ by the following.

$(\mathbf{F}^*)$ $f : [0,T] \times \mathcal{C}_h \to X$ is continuous and satisfies $(\mathbf{F})(2)$. In addition, there exist a function $m \in C([0,T];\mathbb{R}^+)$ and a nondecreasing function $\Psi \in C(\mathbb{R}^+;\mathbb{R}^+)$ such that

$$\|f(t,v)\| \le m(t)\Psi(\|v\|_{\mathcal{C}_h}),\ \forall v \in \mathcal{C}_h.$$

Theorem 4.2. *Assume that the hypotheses* $(\boldsymbol{A})$ *and* $(\boldsymbol{F}^*)$ *hold. If there exists* $R > 0$ *such that*

$$M\|\varphi\|_{\mathcal{C}_h} + MC_\sigma(\|y\| + \sigma\|\varphi\|_{\mathcal{C}_h}) + C_\alpha\Gamma(\alpha)|I_0^{\alpha,\sigma} m|_\infty \Psi(\|\varphi\|_{\mathcal{C}_h} + R) \le R, \tag{4.5}$$

then problem (4.1)-(4.3) *has at least one integral solution.*

Remark 4.1. If Ψ possesses a polynomial growth, then the condition (4.5) takes place provided that $\|(\varphi, y)\|$ as well as $\|I_0^{\alpha,\sigma} m\|_\infty$ are small. In particular, if $\Psi(r) = r^q$ for $q > 1$, then (4.5) is testified with small initial data, that is, the condition on $\|I_0^{\alpha,\sigma} m\|_\infty$ is relaxed.

4.3. FINITE - TIME ATTRACTIVITY

In this section, we prove our main result on finite-time attractivity of zero solution. By $\mathbb{S}(\xi)$, we denote the solution set of the problem (4.1)-(4.3) with respect to initial datum $\xi = (\varphi, y)$. We adopt the following concept of finite-time attractivity of a solution to (4.1).

Definition 4.2. (Finite-time attractivity). Let $\xi_0 = (\varphi_0, y_0) \in \mathcal{C}_h \times X$. The solution $u \in \mathbb{S}(\xi_0)$ is called

(i) attractive on $[0, T]$ if there exists an $\eta > 0$ such that

$$\|v_T - u_T\|_{\mathcal{C}_h} + \|v'(T) - u'(T)\| < \|\varphi - \varphi_0\|_{\mathcal{C}_h} + \|y - y_0\|$$

for all $\xi = (\varphi, y) \in B_\eta(\xi_0) \setminus \{\xi_0\}$ and $v \in \mathbb{S}(\xi)$, where $B_\eta(\xi_0)$ is the closed ball in $\mathcal{C}_h \times X$ centered at ξ_0 with radius η.

(ii) exponentially attractive on $[0, T]$ if

$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \sup_{\xi \in B_\eta(\xi_0)} \sup_{v \in \mathbb{S}(\xi)} (\|v_T - u_T\|_{\mathcal{C}_h} + \|v'(T) - u'(T)\|) < 1.$$

Obviously, exponential attractivity implies attractivity. We give a sufficient condition for the exponential attractivity in following lemma.

Lemma 4.1. *Let $\xi_0 = (\varphi_0, y_0) \in \mathcal{C}_h \times X$ be given. The solution $u \in \mathbb{S}(\xi_0)$ is exponentially attractive on $[0, T]$, provided that*

$$\limsup_{\|\xi\| \to 0} \sup_{v \in \mathbb{S}(\xi + \xi_0)} \frac{\|v_T - u_T\|_{\mathcal{C}_h} + \|v'(T) - u'(T)\|}{\|\xi\|} < 1. \tag{4.6}$$

We first study the attractivity of zero solution to (1).

Lemma 4.2. *Let $(\boldsymbol{A})$ and $(\boldsymbol{F}^*)$ hold with Ψ being locally Lipschitzian and $\Psi(r) = \gamma r + o(r)$ as $r \to 0$, where γ is a nonnegative number. Then*

$$\lim_{\xi = (\varphi, y) \to 0} \sup_{u \in \mathbb{S}(\xi)} \|u_t\|_{\mathcal{C}_h} = 0, \forall t \in [0, T].$$

Remark 4.2. The assumption on behavior of Ψ in Lemma 4.2 holds in the particular case when $f(t, 0) = 0$ and $f(t, \cdot)$ is differentiable at 0 in the sense of Fréchet. In this case, one can choose Ψ as a polynomial.

Theorem 4.3. *Let the hypotheses of Lemma 4.2 hold. Then the zero solution of (1) is exponentially attractive on $[0, T]$, provided that*

$$\Phi(T) + \Sigma(T) < 1 \tag{4.7}$$

where

$$\Phi(t) = M \max\{t, 2 + \sigma t\} e^{-\sigma(t-h)} E_\alpha \left(t^\alpha \gamma |m|_\infty e^{\sigma h} C_\alpha \Gamma(\alpha)\right),$$

and

$$\Sigma(t) = e^{-\sigma t}\max\left\{Ct^{-1} + M\sigma^2 t, M(1+\sigma t)\right\} + \gamma C_\alpha\left(\int_0^t \left[\sigma(t-s)^{\alpha-1} + (t-s)^{\alpha-2}\right] e^{-\sigma(t-s)}\Phi(s)ds\right).$$

In order to prove the attractivity of a nonzero solution, we need the following assumption on the nonlinearity.

($\mathbf{F}^\sharp$) The function $f : [0,T]\times\mathcal{C}_h \to X$ is continuous and satisfies ($\mathbf{F}$)(2). In addition, there exist a function $m \in C([0,T];\mathbb{R}^+)$ and a nondecreasing, locally Lipschitz function $\Psi \in C(\mathbb{R}^+;\mathbb{R}^+)$ such that $\Psi(r) = \gamma r + o(r)$ as $r \to 0$ for some $\gamma \geq 0$, and it holds that

$$\|f(t,v_1) - f(t,v_2)\| \leq m(t)\Psi(\|v_1 - v_2\|_{\mathcal{C}_h}),\ \forall v_1, v_2 \in \mathcal{C}_h.$$

Theorem 4.4. *Let* ($\boldsymbol{A}$) *and* ($\boldsymbol{F}^\sharp$) *hold. Then every solution of* (1) *is exponentially attractive on* $[0,T]$*, provided that*

$$\Phi(T) + \Sigma(T) < 1,$$

where Φ *and* Σ *are the functions given in Theorem 4.3.*

4.4. APPLICATION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following problem

$$\partial_t^{\alpha,\sigma}u(t,x) = \Delta_x u(t,x) + \tilde{f}\left(t, u(t,x), \int_\Omega k(y)u(t-h,y)dy\right),\ x\in\Omega, t\in[0,T], \tag{4.8}$$

$$u(t,x) = 0,\ x\in\partial\Omega, t>0, \tag{4.9}$$

$$u(s,x) = \varphi(s,x), x\in\Omega, s\in[-h,0], \tag{4.10}$$

$$\partial_t u(0,x) = y(x), x\in\Omega, \tag{4.11}$$

where $\alpha\in(1,2), \sigma>0$, $\partial_t^{\alpha,\sigma}$ stands for the tempered Caputo fractional derivative of order α, $\tilde{f} : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $k \in L^2(\Omega)$.

Let $X = L^2(\Omega)$ and $A = \Delta$ with $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$. Then A is the generator of a compact semigroup $\{S(t)\}_{t\geq 0}$. Since the spectrum of $-A$ is the set of eigenvalues of $-A$, which are positive numbers, one sees that $0\in\rho(A)$ and $-A \in \mathsf{Sect}(\theta)$ with $\theta = 0$. So A generates a compact α-resolvent $\{S_\alpha(t)\}_{t\geq 0}$ possessing all properties.

Regarding the nonlinearity $\tilde{f}$, we assume, in addition, that

$$|\tilde{f}(t,y,z)| \leq m(t)|y||z|^p, \forall (t,y,z)\in[0,T]\times\mathbb{R}^2, \tag{4.12}$$

for some $m\in C([0,T];\mathbb{R}^+),\ p>0$.

Put $\mathcal{C}_h = C([-h,0];L^2(\Omega))$. We determine the function $f : [0,T]\times\mathcal{C}_h \to L^2(\Omega)$ as follows

$$f(t,\phi)(x) = \tilde{f}\left(t, \phi(0,x), \int_\Omega k(y)\phi(-h,y)dy\right).$$

Obviously, the problem (4.8)-(4.10) is a prototype of (1)-(2).

By using (4.12), we have

$$\begin{aligned}|f(t,\phi)(x)| &\leq m(t)|\phi(0,x)|\left(\int_\Omega |k(y)||\phi(-\rho(0),y)|dy\right)^p\\ &\leq m(t)|\phi(0,x)|\|k\|_X^p\|\phi(-\rho(0),\cdot)\|_X^p,\end{aligned}$$

thanks to the Hölder inequality. So

$$\begin{aligned}\|f(t,\phi)\| &\leq m(t)\|k\|_X^p\|\phi(0,\cdot)\|_X\|\phi(-\rho(0),\cdot)\|_X^p\\ &\leq m(t)\|k\|_X^p \sup_{s\in[-h,0]}\|\phi(s,\cdot)\|_X^{p+1}\\ &= m(t)\|k\|_X^p\|\phi\|_{\mathcal{C}_h}^{p+1}.\end{aligned}$$

One can check that f is continuous, thanks to the continuity of $\tilde{f}$ and the Lebesgue dominated theorem. Hence f satisfies $(\mathbf{F}^*)$ with $\Psi(r) = \|k\|_X^p r^{p+1} = o(r)$ as $r \to 0$. Using Theorem 4.3, we conclude that the zero solution to (4.8) is exponentially attractive on $[0,T]$, provided that

$$M\max\{T, 2+\sigma T\}e^{\sigma h} + \max\left\{CT^{-1} + M\sigma^2 T, M(1+\sigma T)\right\} < e^{\sigma T},$$

where $M = \sup_{t\in[0,T]}\|S_\alpha(t)\|$ and $C = \sup_{t\in[0,T]}\|tA(g_{\alpha-1} * S_\alpha)(t)\|$.

CONCLUSION

1. RESULTS OF THE THESIS

This thesis has studied the behavior of solutions to some classes of evolution equations and inclusions with delays in Banach space. The results are:

- For a class of fractional partial integro-differential equations with finite delays: We proved the global solvability and the stability of zero solution.
- For a class of fractional partial integro-differential inclusions with infinite delays: We proved the existence of integral solutions and the weak stability of zero solution.
- For semilinear tempered fractional wave equations with finite delays: We proved the existence of solutions on compact intervals even if the nonlinear part may be possibly superlinear, give out definitions on finite-time attractivity and exponential attractivity for system with delays, establish sufficient conditions ensuring finite-time exponential attractivity and then use it to obtain the exponential attractivity on compact intervals of the zero solution.

2. RECOMMENDATION

Beside the results obtained in this thesis, some open questions are:

- Study the behavior of solutions to some fractional differential equations and inclusions without Measure of non-compactness conditions.
- Study the solvability or/and the stability (finite-time or Lyapunov) of solutions for stochastic fractional differential systems.
- Study the solution regularity, the structure of solution set for fractional differential systems.

AUTHOR'S WORKS RELATED TO THE THESIS THAT HAVE BEEN PUBLISHED

1. N.M. Chuong, T.D. Ke, N.N. Quan, *Stability for a class of fractional partial integro-differential equations*, J. Integral Equations Appl. 26 (2014), 145-170. (SCIE)
2. N.T. Anh, T.D. Ke, N.N. Quan. *Weak stability for integro-differential inclusions of diffusion-wave type involving infinite delays*, Discrete Contin. Dyn. Syst. Ser. B 21 (2016), 3637-3654.(SCI)
3. T.D. Ke, N.N. Quan. Finite-time attractivity for semilinear tempered fractional wave equations, *submitted*.

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- Seminar of Department of Analysis, Faculty of Mathematics, Hanoi National University of Eduacation;
- Conference for PHD students, Faculty of Mathematics, Hanoi National University of Eduacation, 2018;